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On the Behaviour of Test Matter in the Vicinity of Singularities

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Abstract

The eikonal approximation of the Klein–Gordon equation in a Riemannian space is considered; this leads to the equations of timelike geodesics. It is shown that, in the vicinity of focal points, the eikonal limit is not valid for test matter focused by gravity. Therefore, first, in the vicinity of singularities considered in the so-called singularity theorems such test matter must be described by their (quantum) field equations and, second, there is no direct physical interpretation of incomplete timelike geodesics.

The results of the theorems established by Hawking & Penrose (1970) and others are often interpreted to mean that the class of space-times considered in these theorems is singular in the sense that incomplete geodesics exist in the space-times in question. Now it is evident that an understanding of these space-times requires a profound physical interpretation; it being especially necessary to give a physical interpretation of the notion 'geodesic incompleteness'.

The purpose of this note is to show that, in general, a direct physical interpretation of timelike incompleteness is impossible because, in the vicinity of focal points considered in the theorems mentioned, those geodesics which are focused there cannot be interpreted as paths of particles.† In particular, it is then senseless to say that the existence of timelike incomplete geodesics means that particles 'disappear'.—This argument supports the viewpoint that a better understanding of singularities and, especially, of the situations described in the cited theorems, requires a local characterization of singularities.

† In crude terms the incompleteness comes about because the space-time has to be 'cut off' to avoid repeated focusing of geodesics. It is physically reasonable to assume that the space-time is cut off in the vicinity of the focal points such that the geodesics end near to these focal points; however, we consider only such situations. This is meant when we speak of 'the vicinity of singularities'.

As matter is described quantum-mechanically by means of the relativistic wave equations of the particles, we consider the general covariantly formulated Klein–Gordon equation

$$\left(\Box - \frac{m^2 c^2}{h^2}\right) A = 0 \tag{1}$$

where $\Box A$ is defined by $\Box A = g^{ik} A_{;ik}$, and discuss the transition from the field equations (1) to the classical limit of particles moving along time-like geodesics.

We start by considering the geometrical-optics approximation (eikonal approximation) (v. Laue, 1961). One obtains this limit from

$$A^i{}_{;i} = 0 \tag{2}$$

$$g^{ik} A_{l;ik} + R_{lk} A^k = 0 \tag{3}$$

(R_{ik} is the Ricci tensor formed out of the background metric g_{ik} and its first and second derivatives), if one studies high-frequency light waves with wavelength λ_0 and assumes that the geometry of the background varies over a distance $L \gg \lambda_0$. Assuming a solution

$$A_l = a_l(x^1, \ldots, x^4) \exp\left[i\omega\phi(x^1, \ldots, x^4)\right] \tag{4}$$

where the orders of R_{ik}, a^l, $a^l{}_{;i}$, l_i, $l_{i;k}$ are given by

$$\begin{gathered} R_{ik} = 0(1), \\ a^l = 0(1), \qquad a^l{}_{;i} = 0(1), \qquad l_i = 0(1), \qquad l_{i;k} = 0(1) \end{gathered} \tag{5}$$

with $l_i = \phi_{,i}$ and $\omega = L/\lambda_0$, by substitution of these relations into (3) and (2) we obtain

$$\omega^2 (l_i l^i a^k) + i\omega(2 l_i a^{k;i} + l^i{}_{;i} a^k) + (\Box a^k + R_i{}^k a^i) = 0 \tag{6}$$

and

$$i\omega l_i a^i + a^i{}_{;i} = 0 \tag{7}$$

When we set terms of the same order equal to zero, we have from (6) and (7)

$$l^i l_i = 0 \tag{8}$$

$$2 a_{k;i} l^i + l^i{}_{;i} a_k = 0 \tag{9}$$

$$l_i a^i = 0, \text{ etc.} \tag{10}$$

Relation (8) is the eikonal equation. Because of $l_{i;k} = l_{k;i}$ it can be rewritten

$$l_{k;i} l^i = 0 \tag{11}$$

This means that, assuming (4), (5) and $L \gg \lambda_0$, the null geodesics can be interpreted as light rays or as paths of photons.

We now investigate the Klein–Gordon equation (1). It will be shown that the transition from (1) to the classical limit of particles moving along timelike geodesics can also be carried out by means of an eikonal approximation.

To this end, we consider the 5-dimensional formulation of (1) given by Klein (1926). We define a 5-dimensional Riemann space with first fundamental form

$$\begin{aligned} d\sigma^2 &\equiv g_{\alpha\beta}\, dx^\alpha\, dx^\beta \\ &= \alpha(dx^0)^2 - g_{ik}\, dx^i\, dx^k \end{aligned} \tag{12}$$

where

$$\alpha = \frac{1}{m^2 c^2} \tag{13}$$

and g_{ik} is the metric of the 4-dimensional space-time. According to Klein, the equation (1) follows from the 5-dimensional wave equation

$$\Box_5\, U(x^0, x^1, \ldots, x^4) = 0 \tag{14}$$

assuming

$$U(x^0, x^1, \ldots, x^4) = A(x^1, \ldots, x^4)\, e^{-i\kappa x^0} \tag{15}$$

($\Box_5$ is the d'Alembert operator with regard to the metric $g_{\alpha\beta}$ and $\kappa \equiv 1/h$).

In order to obtain the classical equation of motion from (14) one can again start by assuming a solution of the eikonal form

$$\begin{aligned} U(x^0, x^1, \ldots, x^4) &= a(x^1, \ldots, x^4) \exp\left[i \frac{\kappa}{\sqrt{\alpha}} S(x^1, \ldots, x^4)\right] \exp(-i\kappa x^0) \\ &= a(x^1, \ldots, x^4) \exp\left[i\kappa\left(\frac{S}{\sqrt{\alpha}} - x^0\right)\right] \end{aligned} \tag{16}$$

Further, we assume, corresponding to (6)

$$\begin{aligned} a &= 0(1), \qquad & a_{,\alpha} &= 0(1) \\ \phi_{,\alpha} &= 0(1), \qquad & \phi_{\alpha;\beta} &= 0(1) \end{aligned} \tag{17}$$

Here the covariant derivative is formed with respect to the metric $g_{\alpha\beta}$ of the 5-dimensional space; for $(\alpha, \beta) = (i, k)$ it is identical with the derivatives defined by means of the g_{ik}; $\phi \equiv (S/\sqrt{\alpha}) - x^0$.

Using the abbreviation

$$\frac{\kappa}{\sqrt{\alpha}} \equiv \omega = (\text{Compton wavelength})^{-1}$$

by substituting (16) into the wave equation (14) we have

$$\omega^2(aS_{,i}\, S^{,i} - a) - i\omega(2a_{,i}\, S^{,i} + aS^i{}_{;i}) - g^{ik} a_{;ik} = 0 \tag{18}$$

With $n_i = S_{,i}$ it follows from (18), instead of (6)

$$\omega^2(an_i\, n^i - a) - i\omega(2a_{,i}\, n^i + an^i{}_{;i}) - \Box a = 0 \tag{19}$$

Assuming that the Compton wavelength $1/\omega = \sqrt{\alpha/\kappa} = h/mc$ of the particles is sufficiently small, we have

$$g^{ik} S_{,i} S_{,k} = -1 \tag{20}$$

and

$$2n^i a_{,i} + an^i{}_{;i} = 0 \tag{21}$$

The equation (20) says that the particles in the classical-mechanics limit travel along timelike geodesics. Equation (21) can be written in the form

$$(a^2 n^i)_{;i} = 0 \tag{22}$$

which may be interpreted as conservation law for the total number of particles.

Let us assume now that, at a great distance from the focal points, the timelike geodesics considered in the Hawking–Penrose theorem can be interpreted as paths of mesons, i.e., we consider a (meson) test matter field such that these geodesics are the result of the eikonal approximation there. Now it can be seen that, in general, such an interpretation of the geodesics being focused is not possible near the focal points because the orders of the terms in (19) are no longer determined by the factors ω^2, ω^1, and ω^0. Indeed, using the Hawking–Penrose result (Hawking & Penrose, 1970) that a goes to infinity as the geodesic parameter approaches the focal points, it can be shown that $\Box a$ can become of the same order as $\omega^2(an^i n_i - a)$.

Using Gaussian coordinates (t,x^1,x^2,x^3), where t is the curve parameter of the timelike geodesics of the hypersurface orthogonal congruence Γ considered in the cited theorem, one can show that the determinant g of the metrical tensor g_{ik} has a zero in t at the focal point of Γ.† Assuming

$$g = G(x^1, x^2, x^3)\, t^n + \dots \tag{23}$$

we obtain

$$\begin{aligned} n^i{}_{;i} &\equiv \frac{1}{\sqrt{-g}} \partial_i(\sqrt{(-g)}\, n^i) \\ &= \frac{n}{2} t^{-1} + \dots \end{aligned} \tag{24}$$

where $n^i = \delta_0{}^i$. With (24) we have from (21)

$$a = A(x^1, x^2, x^3)\, t^{-n/4} + \dots \tag{25}$$

Now with (23) and (25) we get

$$\begin{aligned} \Box a &\equiv \frac{1}{\sqrt{-g}} \partial_i(\sqrt{(-g)}\, g^{ik} a_{,k}) \\ &= -\frac{n}{4\sqrt{-G}} t^{-n/2} \partial_t(\sqrt{(G)}\, At^{n/4-1}) + \dots \\ &\quad + \frac{1}{\sqrt{-G}} t^{-n/2} \partial_\alpha(\sqrt{(G)} \underset{(0)}{g^{\alpha\beta}} A_{,\beta}\, t^{n/4}) + \dots, (\alpha = 1,2,3) \end{aligned} \tag{26}$$

† Details may be found in Borzeszkowski (1973).

Here it is sufficient to remark that there are physically interesting solutions for which the lowest-order term in (26) is given by

$$\sim t^{-n/4-2}$$

(for example, the Kasner solution for $p_3 \neq 1$). That means that, near the focal points, the orders of the terms

$$\omega^2 a(n_i n^i - 1) \qquad \text{and} \qquad -\Box a$$

are given by

$$\begin{aligned} \omega^2 a(n_i n^i - 1) &\sim \frac{1}{\left(\frac{1}{\omega}\right)^2 t^{n/4}} \\ \Box a &\sim \frac{1}{t^{n/4+2}}. \end{aligned} \tag{27}$$

From (27) it is evident that, in general, the considered classical approximation breaks down near the focal points. This breakdown occurs at a parameter distance of the order h/mc.†‡

In summary, in the case of scalar mesons, a direct physical interpretation of 'geodesic incompleteness' is impossible because, in general, geodesics being focused cannot be interpreted as paths of particles in the vicinity of focal points (or of 'singularities'); the quantum field theoretic description of test matter has to be considered there. Such a description of matter requires a local characterization of space-times and their singularities.

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† This corresponds to the optics where the validity of the geometrical-optics approximation breaks down near the focus of lenses (Born & Wolf, 1964). In the case considered above the so-called Raychaudhuri effect acts as gravitational lens.

‡ One could tend to use the answer to the question 'Does the classical approximation break down or not?' as a first step of a classification of focal points.